TRINITY COLLEGE

IN NOMINE DOMINI

Year 11 Mathematics Specialist Units 1&2
Test 6 2022

Calculator Free
Proof & Complex Numbers

STUDENT'S NAME MARKING KEY [KRISZYK]

DATE: Friday 14th October **TIME:** 45 minutes **MARKS**: 48

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser, approved Formula sheet

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

Questions begin on the next page.

1. (7 marks)

Given $w = 3 - 5i$ and $z = i - 2$, evaluate the following:

$-2 + i$

(a) $2w + z$ [1]

$= 2(3-5i) + (-2+i)$

$= 4 - 9i$ ✓

(b) wz [2]

$= (3-5i)(-2+i)$

$= -6 + 13i - 5i^2$ ✓

$= -1 + 13i$ ✓

(c) $\overline{wz}$ [1]

$= -1 - 13i$

✓

(d) $\frac{w}{z}$ [3]

$= \dfrac{3-5i}{-2+i} \times \dfrac{-2-i}{-2-i}$ ✓

$= \dfrac{-6 + 7i + 5i^2}{4 - i^2}$

$= \dfrac{-6 + 7i - 5}{4 + 1}$ ✓

$= \dfrac{-11 + 7i}{5}$ ✓

2. (3 marks)

Determine the complex number w if $w + iw = 1 + 7i$.

let $w = a + bi$

$$a + bi + i(a + bi) = 1 + 7i$$

$$a + bi + ai + bi^2 = 1 + 7i$$

$$(a - b) + (a + b)i = 1 + 7i$$

$a - b = 1$ $\quad\therefore\quad a = 4$

$a + 4 = 7$ $\quad\quad b = 3$

$\therefore \quad w = 4 + 3i$

3. (3 marks)

Express $0.03\bar{4}$ as a rational number.

let $x = 0.03\bar{4}$

$100x = 3.\bar{4}$

$1000x = 34.\bar{4}$

$\therefore \quad 900x = 34.\bar{4} - 3.\bar{4}$

$900x = 31$

$$x = \frac{31}{900}$$

4. (6 marks)

Prove, by contradiction, $\sqrt{7}$ is irrational.

Assume $\sqrt{7}$ is rational, hence $\sqrt{7} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and have no common factors. ✓

$$\therefore \sqrt{7} = \frac{p}{q}$$

$$7 = \frac{p^2}{q^2} \quad ✓$$

$7q^2 = p^2$ $\quad \therefore p^2$ is a multiple of 7

$\rightarrow p$ is a multiple of 7.

$\therefore p = 7k.$ ✓

$$7q^2 = (7k)^2$$

$$7q^2 = 49k^2$$ ✓

$q^2 = 7k^2$ $\quad \therefore q^2$ is a multiple of 7

$\rightarrow q$ is a multiple of 7.

$\therefore q = 7p.$ ✓

$$\therefore \sqrt{7} = \frac{p}{q} = \frac{7k}{7p}$$

$\frac{p}{q}$ has a common factor, hence a contradiction to assumed statement. ✓

$\therefore \sqrt{7}$ is irrational.

5. (4 marks)

Determine all exact solutions (real and complex) for the equation $x^3 - 4x^2 + 13x = 0$

$$x(x^2 - 4x + 13) = 0$$

$$\therefore \quad x = 0 \quad \checkmark$$

$$x^2 - 4x + 13 = 0$$

$$(x-2)^2 + 13 - 4 = 0$$

$$(x-2)^2 + 9 = 0 \quad \checkmark$$

$$x - 2 = \pm\sqrt{-9}$$

$$x - 2 = \pm 3i \quad \checkmark$$

$$x = 2 \pm 3i \quad \checkmark$$

6. (6 marks)

Prove by mathematical induction, $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \quad r \neq 1.$

For $n=1$

$$\text{LHS} = ar^{1-1} = ar^0 = a$$

$$\text{RHS} = \frac{a(\cancel{1-r^1})}{\cancel{1-r}} = a$$

✓

$\therefore$ Case $n=1$ is true.

Assume true for case $n=k$

$$\therefore \quad a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(1-r^k)}{1-r} \quad ✓$$

Case $n=k+1$

$$a + ar^2 + \ldots + ar^{k-1} + ar^{(k+1)-1} = \frac{a(1-r^{k+1})}{1-r} \quad ✓$$

$$\text{LHS} = \frac{a(1-r^k)}{1-r} + ar^k \quad ✓$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r}$$

$$= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r}$$

✓

$$= \frac{a(1-r^{k+1})}{1-r} \quad ✓$$

$$= \text{RHS}$$

$\therefore$ true for $n=k$ and $n=k+1$

$\therefore$ Statement is true.

7. (6 marks)

Consider the expression $m^2 + 7$

(a) Evaluate the expression $m^2 + 7$ for $m = 1, 3, 5, 7$ and 9 [1]

8, 16, 32, 56, 88 ✓

(b) Use your values from (a) to state the largest integer, p, that $m^2 + 7$ is always divisible by, when m is a positive odd integer. [1]

$p = 8$ ✓

(c) Prove that $m^2 + 7$ is always divisible by p when m is a positive odd integer. [4]

let $m = 2n+1$ where $n \geq 0$. ✓ use $2n+1$

$\therefore m^2 + 7 \rightarrow (2n+1)^2 + 7$

$= (2n+1)(2n+1) + 7$

$= 4n^2 + 4n + 8$

$= 4n(n+1) + 8$ ✓

Either n or $n+1$ will be even

$\therefore n(n+1)$ will be even. $\rightarrow 2k$. ✓

$= 4(2k) + 8$

$= 8(k+1)$ ✓

$\therefore m^2 + 7$ is always divisible by p.

8. (13 marks)

(a) Given that $p^n = -i$, where $n \in \mathbb{Z}$, determine each of the following:

(i) p^{n+1} [2]

$= p^n \cdot p^1$ ✓

$= -ip$ ✓

(ii) $(p^n - p^{-n})^2$ [3]

$= (p^n - p^{-n})(p^n - p^{-n})$

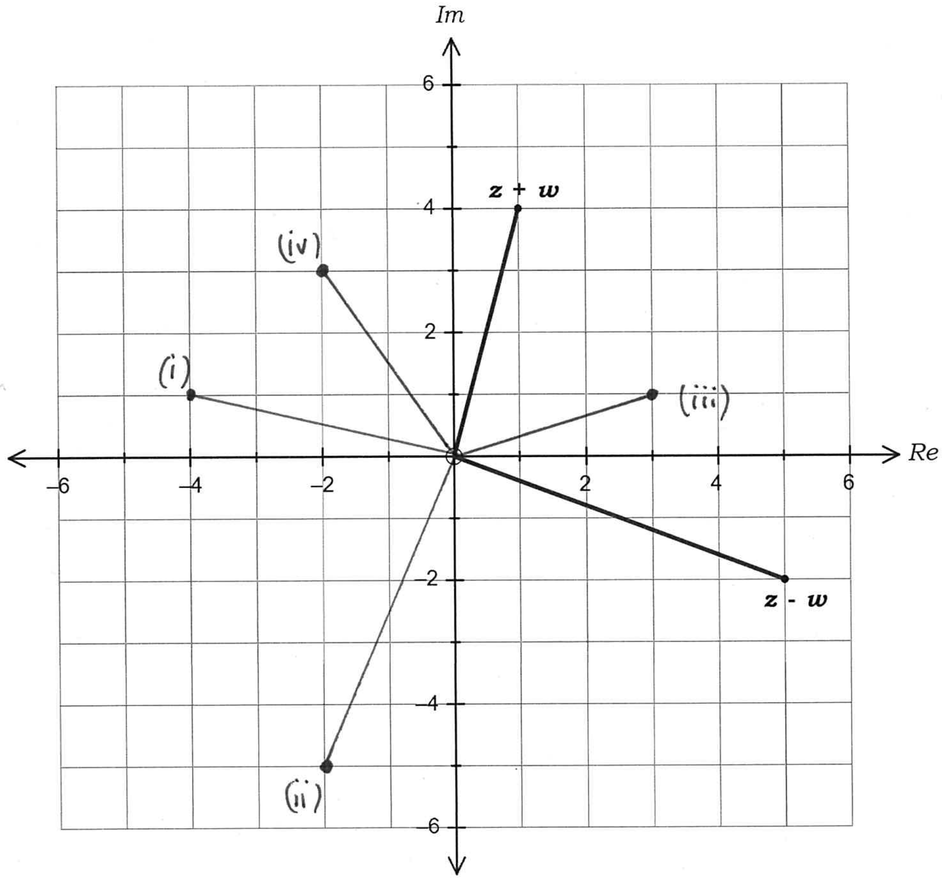
$= (p^n)^2 - 2p^n p^{-n} + (p^{-n})^2$ ✓

$= (-i)^2 - 2p^0 + \left(\frac{1}{i}\right)^2$ ✓

$= -1 - 2 - 1$

$= -4$ ✓

(b) Two complex numbers w and z are such that their addition and subtraction are shown on the diagram below.



Add and label each of the following to the grid above.

(i) $zi + wi$ [2]

$= i(z+w)$

$= i(1+4i)$

$= -4+i$

(ii) $\frac{z-w}{i}$ [2]

$= \frac{5-2i}{i} \times \frac{i}{i}$

$= \frac{5i-2i^2}{i^2}$

$= -2-5i$

(iii) z [2]

$2z = z+w + z-w$

$= 1+4i + 5-2i$

$= 6+2i$

$z = 3+i$

(iv) w [2]

$w = z+w - z$

$= 1+4i - (3+i)$

$w = -2+3i$

Allow FT